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Band Edge Behavior of the Integrated Density of States of Random Jacobi Matrices in Dimension 1

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Let H be a Jacobi matrix acting on $\ell^2(\mathbb{Z})$ and V_ω a random potential of Anderson type. Let $H_\omega = H + V_\omega$. We give a general formula relating the decay of the integrated density of states of H_ω at the edges of the almost sure spectrum of H_ω to the decay of the integrated density of states of H at the edges of the spectrum of H.

KEY WORDS: Random Jacobi matrices; integrated density of states; Lifshitz tails.

0. THE MAIN THEOREM

Let H be a translational invariant Jacobi matrix with exponential off-diagonal decay that is $H = ((h_{k-k'}))_{k,k' \in \mathbb{Z}}$ such that,

- $h_{-k} = \overline{h_k}$ for $k \in \mathbb{Z}$ and for some $k \neq 0$, $h_k \neq 0$.
- there exists $C > 0$ such that, for $k \in \mathbb{Z}$,

$$|h_k| \leqslant C e^{-|k|/C} \tag{1}$$

H defines a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$. Using the Fourier transform, it is easily seen that H is unitarily equivalent to the multiplication by the function $\theta \mapsto h(\theta)$ defined by

$$h(\theta) = \sum_{k \in \mathbb{Z}} h_k e^{ik\theta}$$

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acting as an operator on $L^2([-\pi,\pi])$ (here and in the rest of the paper, we identify the function L^2 over an interval I with the functions that are locally L^2 and periodic of period the length of the interval I; moreover the L^2-norm is normalized so that the constant function 1 has norm 1).

The spectrum of H is $\sigma = h([-\pi,\pi]) = [e^-, e^+]$. We can define the integrated density of states of H and compute

$$n(E) = \frac{1}{2\pi}\,\mu(\{\theta \in [-\pi,\pi];\, h(\theta) \leqslant E\}) \tag{2}$$

where μ denotes the Lebesgue measure on $[-\pi,\pi]$ (see, e.g., ref. 4).

Let V_ω be a diagonal matrix with entries that are independent identically distributed bounded real valued random variables denoted by $(\omega_k)_{k\in\mathbb{Z}}$. Let Ξ be the essential support of the common law of the $(\omega_k)_{k\in\mathbb{Z}}$. Then we can write $\Xi = \bigcup_{j\in J} I_j$ where

- J is an ordered set of indices,
- I_j is the interval $[\omega_j^-, \omega_j^+]$
- for $(j,j')\in J^2$, if $j<j'$ then $\omega_j^- \leqslant \omega_j^+ < \omega_{j'}^- \leqslant \omega_{j'}^+$.

We assume that the $(\omega_k)_{k\in\mathbb{Z}}$ are bounded and that their common law does not decay exponentially at the edges of its support; more precisely, for any $j\in J$, it satisfies

$$\lim_{\varepsilon\to 0} \frac{\log|\log \mathbb{P}\{|\omega-\omega_j^\pm|\leqslant\varepsilon\}|}{-\log\varepsilon} = 0$$

Set $H_\omega = H + V_\omega$; then H_ω defines a bounded ergodic random Jacobi matrix. We denote its almost sure spectrum by Σ and its integrated density of states by $N(E)$. Then, as $\Sigma = [e^-, e^+] + \Xi$ (see, e.g., refs. 4 and 13), we can write $\Sigma = \bigcup_{p=1}^P [E_p^-, E_p^+]$ where the real numbers $(E_p^-, E_p^+)_{1\leqslant p\leqslant P}$ satisfy $E_p^- < E_p^+ < E_{p+1}^- < E_{p+1}^+$. The $(E_p^-)_{1\leqslant p\leqslant P}$ (resp. $(E_p^+)_{1\leqslant p\leqslant P}$) will be called left or lower (resp. right or upper) spectral edges.

Then we have

Theorem 0.1. Under the assumptions made above on H_ω, for $1\leqslant p\leqslant P$, we have

$$\log|\log|N(E)-N(E_p^\pm)|| \underset{\substack{E\to E_p^\pm\\ E\in\Sigma}}{\sim} -\log|n(E-E_p^\pm+e^\pm)-n(e^\pm)| \tag{3}$$

This result calls for some remarks. If H is the discrete Laplace operator then this result is well known even in dimension larger than 1 or for continuous operators (at least at the exterior edges of the spectrum) (see, e.g., refs. 10, 12, 15, and 16 for more references). The main novelty here is that we allow a general class of Jacobi matrices as the kinetic energy of the hamiltonian.

The asymptotic behaviour of the integrated density of states of H near the spectral edges is easy to compute; near the infimum of σ, we get that

$$n(E) \underset{\substack{E \to e^- \\ E > e^-}}{\sim} \alpha(E - e^-)^\rho \tag{4}$$

for some $\alpha > 0$ and $\rho > 0$. An analogous statement holds near the supremum of σ. So that our main theorem says that the integrated density of states of the random operator exhibits a Lifshits tail at its spectral edges. And that the Lifshits exponent is given by the integrated density of states of the underlying operator. By what is known up to now, it seems reasonable to conjecture that the same statement is true for discrete operators in higher dimensions as well as for random Anderson perturbations of periodic Schrödinger operators on $L^2(\mathbb{R}^d)$ (see ref. 9).

The strategy of the proof goes as follows: we first prove that N is well approximated by the expectation value of the integrated density of states of some periodic realisations of H_ω (see Theorem 1.1). We then only need to estimate the integrated density of states of these realisations. The lower bound is obtained by exhibiting an eigenvalue in the relevant energy range for a sufficiently large set of ω. To get the upper bound, we need to understand the Fourier transform of a function localised near $h^{-1}(\{e^\pm\})$. Therefore we use Lemma 3.1.

The fact that $\theta \mapsto h(\theta)$ is real analytic is not of crucial importance. What really matters (to have our method of proof work) is the structure of the set of extremal points of h (and the way h reaches its extremal values). It happens that this structure is extremely simple for analytic functions of one variable. One could imagine to use the same method of proof to get general results about the band edge behaviour for random Schrödinger operators in higher dimension.[9] One of the main obstacles would then be the complicated structure of the set of extremal points of h (even if h is analytic). In this case, it is not clear what is the analogue of Lemma 3.1. In general, even to get the band edge asymptotic behaviour of $n(E)$ is a non-trivial matter (cf. ref. 3 and references therein). In higher dimension, if we assume that the set of extremal points of h consists only of isolated points, our strategy of proof should work. In particular, under this assumption, one should be able to prove that the integrated density of states decays exponentially at the band edges.

If h is not analytic but does not behave too wildly at its extremal points, one can always upper and lower bound it by analytic (or Gevrey class) functions that behave in the same nice way as h. And then perform our analysis on the random hamiltonians generated by these new kinetic energies. This allows us to generalize our result to kinetic energies that do not decrease exponentially off-diagonally.

One could also generalize this study to more general random potentials, for example long range potentials (i.e., with slower decay).

1. PERIODIC APPROXIMATIONS

We will first prove an approximation theorem for the density of states of H_ω. Let $(\omega_j)_{j\in\mathbb{Z}}$ be a realisation of the random variables defined above. Fix $n\in\mathbb{N}^*$. We define the following periodic operator acting on $\ell^2(\mathbb{Z})$

$$H^n_\omega = H + V^n_\omega = H + \sum_{k\in\mathbb{Z}_{2n+1}} \omega_k \sum_{l\in(2n+1)\mathbb{Z}} |\delta_{l+k}\rangle\langle\delta_{l+k}|$$

where $\mathbb{Z}_{2n+1} = \mathbb{Z}/(2n+1)\mathbb{Z}$, $\delta_l = (\delta_{jl})_{j\in\mathbb{Z}}$ (δ_{jl} is the Kronecker symbol) and $|u\rangle\langle u|$ is the orthogonal projection on u a unit vector.

For the H^n_ω, we can define an integrated density of states denoted by N^n_ω (e.g., ref. 14); it is a non-decreasing function that satisfies, for $\varphi\in\mathscr{C}^\infty_0(\mathbb{R})$,

$$(\varphi, dN^n_\omega) = \int_{\mathbb{R}} \varphi(x)\, dN^n_\omega(x) = \frac{1}{2n+1} \sum_{k\in\mathbb{Z}_{2n+1}} \langle \delta_k, \varphi(H^n_\omega)\,\delta_k\rangle \tag{5}$$

Then, we have the

Lemma 1.1. There exists $C>1$ such that, for $\varphi\in\mathscr{C}^\infty_0(\mathbb{R})$, for $k\in\mathbb{N}$ and $n\in\mathbb{N}^*$, we have

$$|\mathbb{E}((\varphi, dN^n_\omega)) - (\varphi, dN)| \leqslant \left(\frac{C}{n}\right)^k \sup_{\substack{x\in\mathbb{R}\\ 0\leqslant j\leqslant k+3}} \left|\frac{d^j\varphi}{d^jx}(x)\right| \tag{6}$$

Proof. The proof follows the lines of the proof of Theorem 5.1 in ref. 10. For the reader's convenience, we reproduce it here. We know (see refs. 4 or 13) that

$$(\varphi, dN) = \int_{\mathbb{R}} \varphi(\lambda)\, dN = \mathbb{E}(\langle \delta_0, \varphi(H_\omega)\,\delta_0\rangle)$$

Averaging (5) and using the fact that the random variables $(\omega_j)_{j\in\mathbb{Z}}$ are i.i.d and the translation invariance of H, we get

$$\mathbb{E}((\varphi, dN^n_\omega)) = \mathbb{E}(\langle \delta_0, \varphi(H^n_\omega)\, \delta_0\rangle)$$

So that we want to estimate $\langle \delta_0, \varphi(H^n_\omega)\, \delta_0\rangle - \langle \delta_0, \varphi(H_\omega)\, \delta_0\rangle$. Therefore we will use Helffer–Sjöstrand's formula[6] that reads

$$\varphi(H_\omega) = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\cdot(z-H_\omega)^{-1}\, dz\wedge d\bar{z} \tag{7}$$

where $\tilde{\varphi}$ is an almost analytic extension of φ (see ref. 11).

So we get that

$$\begin{aligned}
&|\mathbb{E}((\varphi, dN^n_\omega)) - (\varphi, dN)| \\
&\quad\leqslant \frac{1}{4\pi}\,\mathbb{E}\left(\int_{\mathbb{C}}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\right| |\langle \delta_0, ((z-H^n_\omega)^{-1} - (z-H_\omega)^{-1})\, \delta_0\rangle|\, dx\, dy\right) \\
&\quad\leqslant \frac{1}{4\pi}\,\mathbb{E}\left(\int_{\mathbb{C}}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\right| \sum_{|k|\geqslant n} |\omega_k - \omega_{[k]_n}| \right.\\
&\qquad\left.\times |\langle \delta_0, (z-H^n_\omega)^{-1}\, \delta_k\rangle|\, |\langle \delta_k, (z-H_\omega)^{-1}\, \delta_0\rangle|\, dx\, dy\right)
\end{aligned} \tag{8}$$

where $z = x+iy$ and $[k]_n \equiv k \bmod (2n+1)$ (the representant being chosen in $[-n, n]$).

By a Combes–Thomas argument (e.g., ref. 1), we know that there exists $C>1$ such that, uniformly in $(\omega_j)_{j\in\mathbb{Z}}$ and $n\geqslant 1$, we have, for $\operatorname{Im}(z)\neq 0$,

$$|\langle \delta_0, (z-H^n_\omega)^{-1}\, \delta_k\rangle| + |\langle \delta_k, (z-H_\omega)^{-1}\, \delta_0\rangle| \leqslant \frac{C}{|\operatorname{Im}(z)|}\, e^{-|\operatorname{Im}(z)\, k|/C}$$

Hence (8) gives for some $C>1$,

$$|\mathbb{E}((\varphi, dN^n_\omega)) - (\varphi, dN)| \leqslant C\int_{\mathbb{C}}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\right| \frac{1}{|\operatorname{Im}(z)|^3}\, e^{-|\operatorname{Im}(z)\, n|/C}\, dx\, dy$$

Taking into account the properties of almost analytic extensions (cf. ref. 11), we get the announced lemma. ▮

Remark 1.1. The proof of Lemma 1.1 is dimension independent (see ref. 10) and holds for a large class of random operators. We essentially only used the self-adjointness and the off-diagonal exponential fall-off of the Green's kernel at complex energies.

If φ is supported in a open interval I where we have localization in the Aizenman–Molchanov sense (see refs. 1 and 2) then we can choose $\tilde{\varphi}$ supported in a domain D such that for some $C>1$ and $z\in D$ in that domain, we have

$$\mathbb{E}(|\langle \delta_k, (z-H_\omega)^{-1}\,\delta_0\rangle|)\leqslant Ce^{-|k|/C}$$

The estimate given by Lemma 1.1 can then be improved.

As an immediate consequence of Lemma 1.1, we get that $\mathbb{E}(dN^n_\omega)$ converges vaguely to dN. Hence, a classical argument tells us that, except for an at most countable set of energies, N^n_ω converges to N.

The main purpose of Lemma 1.1 is to show that, for $\varepsilon>0$ small, $N(E+\varepsilon)-N(E-\varepsilon)$ is well approximated by $\mathbb{E}(N^n_\omega(E+\varepsilon)-N^n_\omega(E-\varepsilon))$ even when n is only of polynomial size in ε^{-1}. More precisely let φ be a Gevrey class function of Gevrey exponent $\alpha>1$ (see ref. 7); assume moreover that φ has compact support in $(-2,2)$, that $0\leqslant\varphi\leqslant 1$ and that $\varphi\equiv 1$ on $[-1,1]$. Let $E_0\in\mathbb{R}$ and set

$$\varphi_{E_0,\varepsilon}(\cdot)=\varphi\left(\frac{\cdot-E_0}{\varepsilon}\right)$$

Then by Lemma 1.1 and the Gevrey estimates on the derivatives of φ, we get that there exist $C>1$ such that, for $n\geqslant 1$, $k\geqslant 1$ and $0<\varepsilon<1$, we have

$$|\mathbb{E}((\varphi_{E_0,\varepsilon}, dN^n_\omega))-(\varphi_{E_0,\varepsilon}, dN)|\leqslant\varepsilon^{-3}k^{3\alpha}\left(\frac{C}{n\varepsilon}\right)^k k^{k\alpha}$$

We can now optimize the right hand side of the above equation in k and get that, there exist $C>1$ such that, for $n\geqslant 1$ and $0<\varepsilon<1$, we have

$$|\mathbb{E}((\varphi_{E_0,\varepsilon}, dN^n_\omega))-(\varphi_{E_0,\varepsilon}, dN)|\leqslant C(n+\varepsilon^{-1})^3\,e^{-(n\varepsilon/C)^{1/\alpha}+C(n\varepsilon/C)^{-1/\alpha}}$$

Now if we choose $n_\varepsilon=[\varepsilon^{-1-\eta}]$ (where $[\cdot]$ is the entire part of $\cdot$ and $\eta>0$), we get that there exist $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$, we have

$$|\mathbb{E}((\varphi_{E_0,\varepsilon}, dN^{n_\varepsilon}_\omega))-(\varphi_{E_0,\varepsilon}, dN)|\leqslant e^{-\varepsilon^{-\eta/(2\alpha)}}$$

By the definition of φ, as dN^n_ω and dN are positive measures, we have

$$\begin{aligned}\mathbb{E}(N^n_\omega(E_0+\varepsilon)-N^n_\omega(E_0-\varepsilon))&\leqslant\mathbb{E}((\varphi_{E_0,\varepsilon}, dN^n_\omega))\\&\leqslant\mathbb{E}(N^n_\omega(E_0+2\varepsilon)-N^n_\omega(E_0-2\varepsilon))\end{aligned}$$

$$N(E_0+\varepsilon)-N(E_0-\varepsilon)\leqslant(\varphi_{E_0,\varepsilon}, dN)\leqslant N(E_0+2\varepsilon)-N(E_0-2\varepsilon)$$

This gives that for $0 < \varepsilon < \varepsilon_0/2$ we have

$$\begin{aligned}\mathbb{E}(N^{n_\varepsilon}_\omega(E_0+\varepsilon/2) - N^{n_\varepsilon}_\omega(E_0-\varepsilon/2)) - e^{-(\varepsilon/2)^{-\eta/(2\alpha)}} \\ \leqslant N(E_0+\varepsilon) - N(E_0-\varepsilon) \\ \leqslant \mathbb{E}(N^{n_\varepsilon}_\omega(E_0+2\varepsilon) - N^{n_\varepsilon}_\omega(E_0-2\varepsilon)) + e^{-\varepsilon^{-\eta/(2\alpha)}}\end{aligned} \tag{9}$$

If we want to measure exponential decay for N near the band edges, we just have to measure the same exponential decay for $\mathbb{E}(N^{n_\varepsilon}_\omega)$ (where n_ε is chosen as above). This means that when estimating $\mathbb{E}(N^{n_\varepsilon}_\omega)$ the number of eigenvalues (i.e., Floquet eigenvalues) of $H^{n_\varepsilon}_\omega$ in $[E_0-\varepsilon, E_0+\varepsilon]$ will only be polynomial in ε^{-1}; thus it will not be really important for the exponentially small quantity we want to measure. As in the more classical schemes used to prove Lifshits tails (refs. 8, 15, etc.), the double logarithm of the integrated density of states will be well estimated by the double logarithm of the probability to have spectrum in the interval $[E_0-\varepsilon, E_0+\varepsilon]$ for the approximation operator $H^{n_\varepsilon}_\omega$.

To prove Theorem 0.1, in the case of a lower spectral edge, we take $E_0 = E^-_p$ so that, for any admissible ω, $N(E^-_p - \varepsilon) = N^{n_\varepsilon}_\omega(E^-_p - \varepsilon)$ is independent of $\varepsilon > 0$ small enough. Then, if we take $\eta > 0$ large enough so that $\eta/(2\alpha) > \rho$ (where ρ is given by Eq. (4)), to get Theorem 0.1, by (9), we just have to show that

$$\log |\log \mathbb{E}(N^{n_\varepsilon}_\omega(E^-_p+\varepsilon) - N^{n_\varepsilon}_\omega(E^-_p))| \underset{\substack{\varepsilon\to 0\\ \varepsilon>0}}{\sim} -\rho \log \varepsilon \tag{10}$$

This is the purpose of the next section.

2. THE PROOF OF THEOREM 0.1

We will show a lower and an upper bound for the double logarithm of the integrated density of states. The case of the upper edges being dealt with in the same way, we will only give the proof for lower edges of the spectrum.

Let E^-_p be a lower edge of Σ. As $\Sigma = [e^-, e^+] + \Xi$, there exists a unique $j_p \in J$ such that $E^-_p = \omega^-_{j_p} + e^-$ and such that there exists $\delta > 0$ such that for $j' < j_p$, we have $\omega_{j'} + e^+ - \delta \leqslant E^-_p$ (see ref. 13). To simplify the notations, without restricting our purpose, we will assume that $e^- = \omega^-_{j_p} = 0$. So that $E^-_p = 0$.

Let us shortly describe the critical set $Z = h^{-1}(\{0\})$. As h is analytic and not identically 0, Z consists in a finite number of points, i.e., $Z =$

$\{\theta_j;\ 1 \leqslant j \leqslant m\}$ and for any $1 \leqslant j \leqslant m$, there exists $(\rho_j, \alpha_j) \in \mathbb{N}^* \times (0, +\infty)$ such that

$$h(\theta) = \alpha_j(\theta - \theta_j)^{\rho_j}(1 + O(\theta - \theta_j)) \tag{11}$$

This immediately tells us that

$$n(E) \underset{\substack{\varepsilon \to 0 \\ \varepsilon > 0}}{\sim} 2\alpha E^{1/\rho} \tag{12}$$

where

$$\rho = \rho_{j_0} = \sup\{\rho_j;\ 1 \leqslant j \leqslant m\} \qquad \text{and} \qquad \alpha = \sum_{j:\ \rho_j = \rho} \alpha_j^{-1/\rho_j}$$

2.1. Some Floquet Theory

To analyse H_ω^n, we will need some Floquet theory that we develop now. We denote by $\mathscr{F}$: $L^2([-\pi, \pi]) \to \ell^2(\mathbb{Z})$ the usual Fourier series transform. Then, we have, for $u \in L^2([-\pi, \pi])$,

$$\hat{H}_\omega u(\theta) = (\mathscr{F}^* H_\omega \mathscr{F} u)(\theta) = h(\theta)\, u(\theta) + \sum_{j \in \mathbb{Z}} \omega_j (\Pi_j u)(\theta)$$

where

$$(\Pi_j u)(\theta) = \frac{1}{2\pi} e^{ij\theta} \int_{-\pi}^{\pi} e^{-ij\theta} u(\theta)\, d\theta$$

Define the unitary equivalence $U: L^2([-\pi, \pi]) \to L^2([-\pi/(2n+1), \pi/(2n+1)]) \otimes \ell^2(\mathbb{Z}_{2n+1})$ by $(Uu)(\theta) = (u_k(\theta))_{k \in \mathbb{Z}_{2n+1}}$ where the $(u_k(\theta))_{k \in \mathbb{Z}_{2n+1}}$ are defined by

$$u(\theta) = \sum_{k \in \mathbb{Z}_{2n+1}} e^{ik\theta} u_k(\theta) \text{ and the functions } (u_k)_{k \in \mathbb{Z}_{2n+1}} \text{ are } \frac{2\pi}{2n+1} \text{ periodic} \tag{13}$$

Then, for $n > 1$, we compute $U\mathscr{F}^* H_\omega^n \mathscr{F} U^*$ and get that it is the multiplication by the matrix $M_\omega^n(\theta) = ((h_{j-j'}(\theta) + \omega_j \delta_{jj'}))_{(j, j') \in \mathbb{Z}_{2n+1}^2}$ acting on $L^2([-\pi/(2n+1), \pi/(2n+1)]) \otimes \ell^2(\mathbb{Z}_{2n+1})$; here the functions $(h_k)_{k \in \mathbb{Z}_{2n+1}}$ are the components of h decomposed according to (13).

This immediately gives us that the Floquet eigenvalues and eigenvectors of H^n_ω with Floquet quasi-momentum θ (i.e., the solutions of the problem

$$\begin{cases} H^n_\omega u = \lambda u & (\text{where } u = (u_j)_{j\in\mathbb{Z}}) \\ u_{j+k} = e^{-ik\theta} u_j & \text{for} \quad j\in\mathbb{Z}, \quad k\in(2n+1)\,\mathbb{Z}) \end{cases}$$

are nothing but the eigenvalues and eigenvectors (continued quasi periodically) of the $(2n+1)\times(2n+1)$ matrix $M^n_\omega(\theta)$. This gives us that, for $\varepsilon > 0$,

$$N^n_\omega(\varepsilon) - N^n_\omega(0) = \frac{1}{2\pi}\int_{-\pi/(2n+1)}^{\pi/(2n+1)} \sharp\{\text{eigenvalues of } M^n_\omega(\theta) \text{ in } [0, \varepsilon]\}\, d\theta$$

Notice that, as H_ω has no spectrum in $[-\varepsilon, 0)$ for $\varepsilon > 0$ small enough, we know that for any $n > 0$ and almost all ω, $M^n_\omega(\theta)$ also has no spectrum in $[-\varepsilon, 0)$ (cf. refs. 4 and 13); so that

$$N^n_\omega(\varepsilon) - N^n_\omega(0) = \frac{1}{2\pi}\int_{-\pi/(2n+1)}^{\pi/(2n+1)} \sharp\{\text{eigenvalues of } M^n_\omega(\theta) \text{ in } [-\varepsilon, \varepsilon]\}\, d\theta \quad (14)$$

Considering H as being $2n+1$-periodic on $\mathbb{Z}$, we get that the Floquet eigenvalues of H (for the quasi-momentum θ) are $(h(\theta + 2\pi k/(2n+1)))_{k\in\mathbb{Z}_{2n+1}}$ each of them being associated with the vector $(u_k(\theta))_{k\in\mathbb{Z}_{2n+1}}$ having components

$$u_k(\theta) = \frac{1}{\sqrt{2n+1}}\,(e^{-i(\theta + 2\pi k/(2n+1))\,j})_{j\in\mathbb{Z}_{2n+1}}$$

In the sequel, the vectors will be given by their components in this basis. So that if we denote the vectors of the canonical basis by $(v_l(\theta))_{l\in\mathbb{Z}_{2n+1}}$, their components will be

$$v_l(\theta) = \frac{1}{\sqrt{2n+1}}\,(e^{i(\theta + 2\pi k/(2n+1))\,l})_{k\in\mathbb{Z}_{2n+1}}$$

We define the vectors $(v_l)_{l\in\mathbb{Z}_{2n+1}}$ by

$$v_l = e^{-il\theta} v_l(\theta) = \frac{1}{\sqrt{2n+1}}\,(e^{i(2\pi kl/(2n+1))})_{k\in\mathbb{Z}_{2n+1}}$$

2.2. The Lower Bound

By (14) and (10), for $|\theta| \leqslant \pi/(2n+1)$ we just need to prove the right lower bound for the probability that $M^n_\omega(\theta)$ has an eigenvalue in $[-\varepsilon, \varepsilon]$. We will do this by explicitly constructing such an eigenvalue for a sufficiently large set of ω's.

Let $a \in \ell^2(\mathbb{Z}_{2n+1})$ be expressed by its coordinates in the basis $(u_k(\theta))_{k \in \mathbb{Z}_{2n+1}}$, i.e., $a = \sum_{k \in \mathbb{Z}_{2n+1}} a_k u_k(\theta)$. Then

$$\|M^n_\omega(\theta)\, a\|^2_{\ell^2(\mathbb{Z}_{2n+1})} = \sum_{k \in \mathbb{Z}_{2n+1}} h^2\left(\theta + \frac{2k\pi}{2n+1}\right) |a_k|^2 + \frac{1}{2n+1} \sum_{l \in \mathbb{Z}_{2n+1}} \omega_l^2\, |A_l|^2 \tag{15}$$

where

$$A_l = \sum_{k \in \mathbb{Z}_{2n+1}} a_k e^{-2i\pi kl/(2n+1)} = \sqrt{2n+1}\, \langle a, v_l \rangle_{\ell^2(\mathbb{Z}_{2n+1})}$$

On A_l, we perform q discrete integration by parts ($q \in \mathbb{N}$) that is

$$\begin{aligned} A_l &= \frac{1}{(e^{-2i\pi l/(2n+1)} - 1)^q} \sum_{k \in \mathbb{Z}_{2n+1}} a_k \sum_{p=0}^{q} (-1)^p \binom{p}{q} e^{-2i\pi(k+p)l/(2n+1)} \\ &= \frac{1}{(e^{-2i\pi l/(2n+1))} - 1)^q} \sum_{k \in \mathbb{Z}_{2n+1}} e^{-2i\pi kl/(2n+1)} \sum_{p=0}^{q} (-1)^p \binom{p}{q} a_{k-p} \end{aligned}$$

Pick a function $a \in \mathscr{C}_0^\infty((-1/2, 1/2)) \subset L^2([-1/2, 1/2])$ and set the components of the vector v defined above to be $a_k = (1/\sqrt{2n+1})\, a(k/(2n+1))$. Then

$$\sum_{p=0}^{q} (-1)^p \binom{p}{q} a_{k-p} = \frac{1}{\sqrt{2n+1}} ((1-\tau_n)^q a)\left(\frac{k}{2n+1}\right)$$

where $(\tau_n a)(x) = a(x - 1/(2n+1))$. So that

$$|A_l|^2 \leqslant \frac{1}{(2n+1)^{2q}\, |e^{-2i\pi l/(2n+1)} - 1|^{2q}} \sup_{x \in (-1/2, 1/2)} |a^{(q)}(x)|^2$$

Let $a \in \mathscr{C}_0^\infty((-1/4, 1/4)) \subset L^2([-1/2, 1/2])$, a non negative such that $\|a\|_{L^2([-1/2, 1/2])} = 1$. Pick $C > 1$ a constant. Set $a^\varepsilon(\cdot) = (C\varepsilon)^{-1/2p} \times a((C\varepsilon)^{-1/p}(\cdot - \theta_0))$ and $n = n_\varepsilon = [\varepsilon^{-1-\eta}]$ where η is chosen as above large enough and θ_0 is a point in Z that gives a main contribution to (12).

We define the a_k^ε as above; using Riemann sums and the fact that $\|a^\varepsilon\|_{L^2([-1/2,1/2])} = \|a\|_{L^2([-1/2,1/2])} = 1$, we compute

$$\sum_{k \in \mathbb{Z}_{2n+1}} |a_k^\varepsilon|^2 = 1 + O(\|a\| \, \|a'\| \, \varepsilon^{\eta - 1/\rho}) \tag{17}$$

As $h(\theta) \leqslant \alpha(\theta - \theta_0)^\rho$ for θ close to θ_0, we get that

$$\sum_{k \in \mathbb{Z}_{2n+1}} h^2\left(\theta + \frac{2k\pi}{2n+1}\right) |a_k^\varepsilon|^2 \leqslant 4\alpha^2\varepsilon^2/C \leqslant \varepsilon^2/3 \tag{18}$$

if C is large enough and ε small enough.

Set $L_\varepsilon = [\varepsilon^{-1/\rho - \nu}]$, here ν will be chosen small. The second term in the right hand side of (15) can be split into

$$\sum_{l \in \mathbb{Z}_{2n+1}} \omega_l^2 \, |A_l^\varepsilon|^2 = \sum_{|l| \leqslant L_\varepsilon} \omega_l^2 \, |A_l^\varepsilon|^2 + \sum_{1 + L_\varepsilon \leqslant |l| \leqslant n} \omega_l^2 \, |A_l^\varepsilon|^2 = O^- + O^+$$

To deal with O^+, as the ω are bounded, we use the integration by parts (16) to get

$$\begin{aligned}
O^+ &\leqslant K \sum_{1 + L_\varepsilon \leqslant |l| \leqslant n} \frac{(C\varepsilon)^{-(2q+1)/\rho}}{(2n+1)^{2q} \, |e^{-2i\pi l/(2n+1)} - 1|^{2q}} \, \|a^{(q)}\|_\infty^2 \\
&\leqslant K(2n+1) \, \|a^{(q)}\|_\infty^2 \sup_{1 + L_\varepsilon \leqslant |l| \leqslant n} \left(\frac{(C\varepsilon)^{-1/\rho}}{(2n+1) \, |e^{-2i\pi l/(2n+1)} - 1|}\right)^{2q} (C\varepsilon)^{-1/\rho} \\
&\leqslant K(2n+1) \, \|a^{(q)}\|_\infty^2 \, (C\varepsilon)^{-1/\rho} \left(\frac{(C\varepsilon)^{-1/\rho}}{L_\varepsilon}\right)^{2q} \\
&\leqslant K(2n+1) \, \|a^{(q)}\|_\infty^2 \, (C\varepsilon)^{-1/\rho} \, (C\varepsilon^\nu)^{2q}
\end{aligned}$$

so that for q large enough depending ν, ρ and C, we get, for ε small enough,

$$O^+ \leqslant (2n+1) \, \varepsilon/3 \tag{19}$$

Now, if we choose $|\omega_l| \leqslant \varepsilon/3$ for $|l| \leqslant L_\varepsilon$ then, by (17),

$$O^- \leqslant \varepsilon^2/9 \sum_{l \in \mathbb{Z}_{2n+1}} |A_l^\varepsilon|^2 \leqslant (2n+1) \, \varepsilon^2/9 \sum_{l \in \mathbb{Z}_{2n+1}} |a_l^\varepsilon|^2 \leqslant (2n+1) \, \varepsilon^2/3$$

for ε small enough.

Adding this last equation to (18) and (19), we get that, if $|\omega_l| \leqslant \varepsilon/3$ for $|l| \leqslant L_\varepsilon$, then

$$|M^{n_\varepsilon}_\omega(0)\, a^\varepsilon|^2_{\ell^2(\mathbb{Z}_{2n+1})} \leqslant \varepsilon^2$$

Hence, for ε small enough,

$$(\mathbb{P}(|\omega| \leqslant \varepsilon/3))^{2L_\varepsilon+1} \leqslant \mathbb{E}(N^{n_\varepsilon}_\omega(\varepsilon))$$

Using (9), taking the log log, dividing by $\log \varepsilon$ and taking the limit $\varepsilon \to 0$, we get

$$\liminf_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{\log |\log(N(\varepsilon) - N(0))|}{\log \varepsilon} \geqslant -\frac{1}{\rho} - \nu$$

This holds for any $\nu > 0$ hence gives the expected lower bound.

2.3. The Upper Bound

Using (14) and Fubini's theorem, we have

$$\mathbb{E}(N^{N_\varepsilon}_\omega(\varepsilon)) \leqslant (2N_\varepsilon + 1) \int_{-\pi/(2N_\varepsilon+1)}^{\pi/(2N_\varepsilon+1)} \mathbb{P}(\Omega_\varepsilon(\theta))\, d\theta. \tag{21}$$

where $\Omega_\varepsilon(\theta)$ is the event {there exists an eigenvalue of $M^{N_\varepsilon}_\omega(\theta)$ in $[-\varepsilon, \varepsilon]$}. So, if we choose N_ε to be a polynomial (of sufficiently large degree) in ε^{-1}, we just have to upper bound the probability of $\Omega_\varepsilon(\theta)$ uniformly in θ. To do this we will first estimate this event by an analogous event for a new random operator that is non-negative. The gain will then be that we will be working at the bottom of the spectrum of this new operator. To do this we follow the ideas developed in Section 5.3 of ref. 10.

For $t \in [0, 1]$, the random operator $tH + V_\omega$ has an almost sure spectrum denoted by Σ_t. Then, for $t \in [0, 1]$, $\Sigma_t \subset \Sigma$ (as $\sigma = [0, e^+]$). So that for $\varepsilon > 0$ small enough, $\Sigma_t \cap [-\varepsilon, 0) = \varnothing$. Hence, for any $n > 1$, $\theta \in [-\pi/(2n+1), \pi/(2n+1))]$ and almost all ω, $M^n_\omega(t, \theta)$ has no spectrum in $[-\varepsilon, 0)$ (here $M^n_\omega(t, \theta)$ is the matrix $((th_{j-j'}(\theta) + \omega_j \delta_{jj'}))_{(j,j') \in \mathbb{Z}^2_{2n+1}}$ associated to $tH + V_\omega$). Thus we get that, for $\varepsilon > 0$ small enough,

$$\begin{aligned}\#\{\text{eigenvalues of } M^n_\omega(\theta) \text{ less than } -\varepsilon\} &= \#\{k \in \mathbb{Z}_{2n+1};\, \omega_k < -\varepsilon\} \\ &= \#\{k \in \mathbb{Z}_{2n+1};\, \omega_k < 0\}\end{aligned}$$

Let Π_ω^+ (resp. Π_ω^-) be the orthogonal projector on the sites k such that $\omega_k < 0$ (resp. on the sites k such that $\omega_k < 0$) that is $\Pi_\omega^+ = \sum_{k \in \mathbb{Z}_{2n+1}, \omega_k > 0} |\delta_k\rangle\langle\delta_k|$ and $\Pi_\omega^- = \sum_{k \in \mathbb{Z}_{2n+1}, \omega_k < 0} |\delta_k\rangle\langle\delta_k|$ (acting on $\ell^2(\mathbb{Z}_{2n+1})$). Then using the minimax principle (see Lemma 5.3 in ref. 10), we get that, ω almost surely,

$$\begin{aligned} &\#\{\text{eigenvalues of } M_\omega^n(\theta) \text{in } [-\varepsilon, \varepsilon]\} \\ &\qquad \leqslant \#\{\text{eigenvalues of } \Pi_\omega^+ M_\omega^n(\theta) \Pi_\omega^+ \text{ less than } \varepsilon\} \end{aligned} \tag{22}$$

where $\Pi_\omega^+ M_\omega^n(\theta) \Pi_\omega^+$ is acting on $\Pi_\omega^+(\ell^2(\mathbb{Z}_{2n+1}))$ and the multiplicity of the eigenvalues is counted on this space. Define a set of new random variables $(\tilde{\omega}_k)_{k \in \mathbb{Z}}$ as follows

$$\tilde{\omega}_k = \begin{cases} \omega_k & \text{if} \quad \omega_k \geqslant 0 \\ \text{ess-sup}(\omega_k) & \text{if not} \end{cases}$$

Notice that the random variables $(\tilde{\omega}_k)_{k \in \mathbb{Z}}$ and $(\omega_k)_{k \in \mathbb{Z}}$ have the same law near 0. Moreover the $(\tilde{\omega}_k)_{k \in \mathbb{Z}}$ have a positive expectation.

Then, if we define $M_{\tilde{\omega}}^n(\theta) = ((h_{j-j'}(\theta) + \tilde{\omega}_j \delta_{jj'}))_{(j,j') \in \mathbb{Z}_{2n+1}^2}$, we have $\Pi_\omega^+ M_{\tilde{\omega}}^n(\theta) \Pi_\omega^+ = \Pi_\omega^+ M_\omega^n(\theta) \Pi_\omega^+$. Thus, by (22), we get that, ω almost surely

$$\begin{aligned} &\#\{\text{eigenvalues of } M_\omega^n(\theta) \text{ in } [-\varepsilon, \varepsilon]\} \\ &\qquad \leqslant \#\{\text{eigenvalues of } \Pi_\omega^+ M_{\tilde{\omega}}^n(\theta) \Pi_\omega^+ \text{ less than } \varepsilon\} \\ &\qquad \leqslant \#\{\text{eigenvalues of } M_{\tilde{\omega}}^n(\theta) \text{ less than } \varepsilon\} \end{aligned}$$

(here the eigenvalues of $\Pi_\omega^+ M_{\tilde{\omega}}^n(\theta) \Pi_\omega^+$ are counted (with their multiplicity) for the operator considered as acting on $\Pi_\omega^+(\ell^2(\mathbb{Z}_{2n+1}))$).

We define $V_{\tilde{\omega}}$ to be the diagonal matrix having the $(\tilde{\omega}_k)_{k \in \mathbb{Z}}$ as diagonal entries and $H_{\tilde{\omega}} = H + V_{\tilde{\omega}}$. Then $M_{\tilde{\omega}}^n(\theta)$ is associated to $H_{\tilde{\omega}}$ in the same way as $M_\omega^n(\theta)$ is associated to H_ω. Notice that $H_{\tilde{\omega}}$ is non-negative and has 0 in its almost sure spectrum.

So we now only need to estimate the probability of the event $\tilde{\Omega}_\varepsilon(\theta) = \{$there exists an eigenvalue of $M_{\tilde{\omega}}^{N_\varepsilon}(\theta)$ less than $\varepsilon\}$. To simplify the notations, we will forget about the superscript $\tilde{}$ i.e., we will denote $\tilde{\Omega}_\varepsilon(\theta)$ by $\Omega_\varepsilon(\theta)$ and $\tilde{\omega}$ by ω.

We recall that $Z = h^{-1}(\{0\}) = \{\theta_1, ..., \theta_m\}$ and that ρ is the largest of the orders of these zeroes of h. For $\eta > 0$ and $1/\rho > 2\nu > 0$, define

- $2L'_\varepsilon + 1 = [\varepsilon^{-1/\rho + 2\nu}]_o$,
- $2K'_\varepsilon + 1 = [\varepsilon^{-1-\eta+1/\rho-\nu}]_o [\varepsilon^{-\nu}]_o$,
- $2N_\varepsilon + 1 = (2K'_\varepsilon + 1)(2L'_\varepsilon + 1)$

(where $[\cdot]_o$ denotes the largest odd integer smaller than $\cdot$).

Then we have the

Lemma 2.1. For any $1 > \delta > 0$ and any $1/\rho > 2\nu > 0$, there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, we have, for $|\theta| \leqslant \pi/(2N_\varepsilon + 1)$,

$$\Omega_\varepsilon(\theta) \subset \bigcup_{|k'| \leqslant K'_\varepsilon} \left(\bigcup_{1 \leqslant j < j' \leqslant m} \Omega_\delta^{j, j', k'} \bigcup \Omega_\delta^{k'} \right) \tag{23}$$

where, for $1 \leqslant j < j' \leqslant m$ and $|k'| \leqslant K'_\varepsilon$, we define the events

$$\Omega_\delta^{j, j', k'} = \left\{ \omega; \frac{1}{2L'_\varepsilon + 1} \left| \sum_{|l'| \leqslant L'_\varepsilon} \omega_{k'(2K'_\varepsilon + 1) + l'} \, e^{i(\theta_j - \theta_{j'}) l'} \right| \geqslant \delta \right\}$$

$$\Omega_\delta^{k'} = \left\{ \omega; \frac{1}{2L'_\varepsilon + 1} \sum_{|l'| \leqslant L'_\varepsilon} \omega_{k'(2K'_\varepsilon + 1) + l'} \leqslant \delta \right\}$$

Before proving this lemma, let us use it to end the proof of Theorem 0.1. Therefore we just need to estimate the probability of the events $\Omega_\delta^{j, j', k'}$ and $\Omega_\delta^{k'}$. We pick $\eta > 0$ as in (9) and ν small. The random variables $(\omega_n)_{n \in \mathbb{Z}}$ are i.i.d, non-negative. Let $\bar{\omega} > 0$ be their common expectation value; so for $0 < \delta < \bar{\omega}$, by classical large deviation estimates (cf. ref. 5), we know that, for some $c > 0$, $|k'| \leqslant K'_\varepsilon$, we have

$$\mathbb{P}(\Omega_\delta^{k'}) \leqslant e^{-cL'_\varepsilon} \tag{24}$$

For $\omega \in \Omega_\delta^{j, j', k'}$, as $\theta_j \neq \theta_{j'}$, for some $C > 0$, we have

$$\frac{1}{2L'_\varepsilon + 1} \left| \sum_{|l'| \leqslant L'_\varepsilon} (\omega_{k'(2K'_\varepsilon + 1) + l'} - \bar{\omega}) \, e^{i(\theta_j - \theta_{j'}) l'} \right| \geqslant \delta - \frac{C}{2L'_\varepsilon + 1} \geqslant \delta/2$$

for ε small enough. The random variables $(\omega_{k'(2K'_\varepsilon + 1) + l'} - \bar{\omega}) \, e^{i(\theta_j - \theta_{j'}) l'}$ are independent and have expectation value 0. By classical large deviation estimates, we get that, for some $c > 0$,

$$\mathbb{P}(\Omega_\delta^{j, j', k'}) \leqslant e^{-cL'_\varepsilon}$$

We add this to (24), plug it into (21) then into (9); we take the log log, divide by $\log \varepsilon$ and let ε tend to 0 to get

$$\limsup_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{\log |\log N(\varepsilon)|}{\log \varepsilon} \leqslant -\frac{1}{\rho} + \nu$$

But this holds for any $\nu > 0$ small enough. Taking (20) into consideration, this ends the proof of Theorem 0.1.

Proof of Lemma 2.1. To simplify the notations, we will forget about the ε subscript and set $N = N_\varepsilon$ and so on. We define

- $2K + 1 = [\varepsilon^{-1-\eta+1/\rho-\nu}]_o$,
- $2L + 1 = [\varepsilon^{-1/\rho+2\nu}]_o [\varepsilon^{-\nu}]_o$.

Pick $\omega \in \Omega_\varepsilon(\theta)$ so there exists $a = \sum_{k \in \mathbb{Z}_{2n+1}} a_k u_k(\theta)$ such that

- $\|a\|_{\ell^2(\mathbb{Z}_{2N+1})} = \sqrt{\sum_{k \in \mathbb{Z}_{2N+1}} |a_k|^2} = 1$,
- $\langle M_\omega^N(\theta)\, a, a \rangle_{\ell^2(\mathbb{Z}_{2N+1})} \leqslant \varepsilon$,
- v_l is defined above.

For $1 \leqslant j \leqslant m$ and $\theta \in [-\pi/(2N+1), \pi/(2N+1)]$, we know that, for some $C > 0$ and for ε small enough

$$\left| \frac{2\pi k}{2N+1} - \theta_j \right| \geqslant \frac{1}{2L+1} \implies \left(h\left(\theta + \frac{2\pi k}{2N+1}\right) \geqslant \varepsilon^{1-\nu\rho}/C \right) \tag{25}$$

For $1 \leqslant j \leqslant m$, we define $k_j = [((2N+1)\,\theta_j)/2\pi]$ and a^j by

$$(a^j)_k = \begin{cases} a_k & \text{if } |k - k_j| \leqslant K \\ 0 & \text{if not} \end{cases}$$

The $(a^j)_{1 \leqslant j \leqslant m}$ are pairwise orthogonal. By (25) and as V_ω^N is non-negative, we get that

$$\left\| a - \sum_{j=1}^m a^j \right\|^2_{\ell^2(\mathbb{Z}_{2N+1})} \leqslant C\varepsilon^{\nu\rho} \leqslant C\varepsilon^{2\nu} \tag{26}$$

We can then normalise $\sum_{j=1}^m a^j$ and get that, for some $C > 0$ and ε small enough

$$\left\langle M_\omega^N(\theta) \left(\sum_{j=1}^m a^j \right), \left(\sum_{j=1}^m a^j \right) \right\rangle_{\ell^2(\mathbb{Z}_{2N+1})} \leqslant C\varepsilon^\nu$$

This in particular implies

$$\sum_{j=1}^{m} \sum_{l \in \mathbb{Z}_{2N+1}} \omega_l |\langle a^j, v_l \rangle|^2 + 2\,\mathrm{Re}\left(\sum_{1 \leqslant j < j' \leqslant m} \sum_{l \in \mathbb{Z}_{2N+1}} \omega_l \langle a^j, v_l \rangle \overline{\langle a^{j'}, v_l \rangle} \right) \leqslant C\varepsilon^{\nu} \tag{27}$$

We now translate each of the a^j by k_j so as to centre its support at 0. The vector thus obtained we again call a^j; then (27) becomes

$$\sum_{j=1}^{m} \sum_{l \in \mathbb{Z}_{2N+1}} \omega_l |\langle a^j, v_l \rangle|^2 + 2\,\mathrm{Re}\left(\sum_{1 \leqslant j < j' \leqslant m} \sum_{l \in \mathbb{Z}_{2N+1}} e^{(2i\pi(k_j - k_{j'})\,l)/(2N+1)} \omega_l \langle a^j, v_l \rangle \overline{\langle a^{j'}, v_l \rangle} \right) \leqslant C\varepsilon^{\nu}$$

We now apply Lemma 3.1 to each a^j; observing that all the operators involved are bounded, that m is fixed and that in our case K/K' behaves like ε^{ν}, we get that, for some $C > 0$ and ε small enough

$$\sum_{j=1}^{m} \sum_{l \in \mathbb{Z}_{2N+1}} \omega_l |\langle \tilde{a}^j, v_l \rangle|^2 + 2\,\mathrm{Re}\left(\sum_{1 \leqslant j < j' \leqslant m} \sum_{l \in \mathbb{Z}_{2N+1}} e^{(2i\pi(k_j - k_{j'})\,l)/(2N+1)} \omega_l \langle \tilde{a}^j, v_l \rangle \overline{\langle \tilde{a}^{j'}, v_l \rangle} \right) \leqslant C\varepsilon^{\nu}$$

So that

$$\begin{aligned} &\sum_{j=1}^{m} \sum_{k' \in \mathbb{Z}_{2K'+1}} \left(\frac{1}{2L'+1} \sum_{l' \in \mathbb{Z}_{2L'+1}} \omega_{l' + k'(2L'+1)} \right) (2L'+1)\, |\langle \tilde{a}^j, v_{k'(2L'+1)} \rangle|^2 \\ &\quad + \sum_{\substack{1 \leqslant j < j' \leqslant m \\ k' \in \mathbb{Z}_{2K'+1}}} 2\,\mathrm{Re}(S(j, j', k')\, e^{(2i\pi(k_j - k_{j'})\,k')/(2K'+1)} \\ &\quad \times \langle \tilde{a}^j, v_{k'(2L'+1)} \rangle \overline{\langle \tilde{a}^{j'}, v_{k'(2L'+1)} \rangle}) \leqslant C\varepsilon^{\nu} \end{aligned}$$

where

$$S(j, j', k') = \frac{1}{2L'+1} \sum_{l' \in \mathbb{Z}_{2L'+1}} \omega_{l' + k'(2L'+1)}\, e^{(2i\pi(k_j - k_{j'})\,l')/(2N+1)}$$

Hence

$$\sum_{j=1}^{m} \sum_{k' \in \mathbb{Z}_{2K'+1}} \left(\frac{1}{2L'+1} \sum_{l' \in \mathbb{Z}_{2L'+2}} \omega_{l'+k'(2L'+1)} \right) (2L'+1) \, |\langle \tilde{a}^j, v_{k'(2L'+1)} \rangle|^2$$
$$\leqslant 2 \sum_{\substack{1 \leqslant j < j' \leqslant m \\ k' \in \mathbb{Z}_{2K'+1}}} |S(j, j', k')| \, (2L'+1) \, |\langle \tilde{a}^j, v_{k'(2L'+1)} \rangle \langle \tilde{a}^{j'}, v_{k'(2L'+1)} \rangle| + C\varepsilon^{\nu} \tag{28}$$

If we define

$$\Sigma(j, j', k') = \frac{1}{2L'+1} \sum_{l' \in \mathbb{Z}_{2L'+1}} \omega_{l'+k'(2L'+1)} e^{i(\theta_j - \theta_{j'}) l'}$$

as $|2i\pi k_j/(2N+1) - \theta_j| \leqslant 1/(2N+1)$, we get that

$$|\Sigma(j, j', k') - S(j, j', k')| = O(\varepsilon^{1+\eta-1/\rho-2\nu})$$

So, as $\|\tilde{a}^j\| \leqslant 2$ (for ε small enough by Lemma 3.1), by (28), we obtain

$$\sum_{j=1}^{m} \sum_{k' \in \mathbb{Z}_{2K'+1}} \left(\frac{1}{2L'+1} \sum_{l' \in \mathbb{Z}_{2L'+1}} \omega_{l'+k'(2L'+1)} \right) (2L'+1) \, |\langle \tilde{a}^j, v_{k'(2L'+1)} \rangle|^2$$
$$\leqslant 2 \sum_{\substack{1 \leqslant j < j' \leqslant m \\ k' \in \mathbb{Z}_{2K'+1}}} |\Sigma(j, j', k')| \, (2L'+1) \, |\langle \tilde{a}^j, v_{k'(2L'+1)} \rangle \langle \tilde{a}^{j'}, v_{k'(2L'+1)} \rangle| + C\varepsilon^{\nu} \tag{29}$$

Pick $\delta > 0$. Then,

- either, for some $j \neq j'$ and $k' \in \mathbb{Z}_{2K'+1}$, we have

$$\left| \frac{1}{2L'+1} \sum_{l' \in \mathbb{Z}_{2L'+1}} \omega_{l'+k'(2L'+1)} e^{i(\theta_j - \theta_{j'}) l'} \right| \geqslant \delta$$

then $\omega \in \Omega_{\delta}^{j, j', k'}$.

- or for all $j \neq j'$ and $k' \in \mathbb{Z}_{2K'+1}$, we have

$$\left| \frac{1}{2L'+1} \sum_{l' \in \mathbb{Z}_{2L'+1}} \omega_{l'+k'(2L'+1)} e^{i(\theta_j - \theta_{j'}) l'} \right| \leqslant \delta$$

In this case, (29) tells us that, for ε small enough,

$$\sum_{j=1}^{m} \sum_{k' \in \mathbb{Z}_{2K'+1}} \left(\frac{1}{2L'+1} \sum_{l' \in \mathbb{Z}_{2L'+1}} \omega_{l'+k'(2L'+1)} \right)$$
$$\times (2L'+1) \, |\langle \tilde{a}^j, v_{k'(2L'+1)} \rangle|^2 \leqslant 4m\delta \tag{30}$$

As a is normalised, by (26) and Lemma 3.1, for ε small enough, we have

$$\sum_{j=1}^{m} \sum_{k' \in \mathbb{Z}_{2K'+1}} (2L'+1) \, |\langle \tilde{a}^j, v_{k'(2L'+1)} \rangle|^2 \geqslant 1/2$$

So that (30) implies that, for some $k' \in \mathbb{Z}_{2K'+1}$,

$$\frac{1}{2L'+1} \sum_{l' \in \mathbb{Z}_{2L'+1}} \omega_{l'+k'(2L'+1)} \leqslant 8m\delta$$

Hence ω belongs $\Omega^{k'}_{8m\delta}$ for some k'.

As we can choose δ as small as we want, this ends the proof of Lemma 2.1.

3. APPENDIX: THE KEY LEMMA

The lemma, key to our analysis, roughly says that if a is a vector in $\ell^2(\mathbb{Z}_{2N+1})$ with coefficients concentrated near 0 in a region of width $K < N$, then up to a small error in ℓ^2-norm, a can be considered to have Fourier coefficients that are constant over intervals of length N/K.

More precisely we have

Lemma 3.1. Assume N, L, K, K' L' are positive integers such that

- $2N+1 = (2K+1)(2L+1) = (2K'+1)(2L'+1)$ such
- $K < K'$ and $L' < L$.

Pick $a = (a_n)_{n \in \mathbb{Z}_{2N+1}} \in \ell^2(\mathbb{Z}_{2N+1})$ such that,

$$\text{for} \quad |n| > K, \quad a_n = 0$$

Then there exists $\tilde{a} \in \ell^2(\mathbb{Z}_{2N+1})$ such that

1. $\|a - \tilde{a}\|_{\ell^2(\mathbb{Z}_{2N+1})} \leqslant C_{K,K'} \, \|a\|_{\ell^2(\mathbb{Z}_{2N+1})}$ where $C_{K,K'} \sim_{K/K' \to 0} \pi K/K'$.
2. for $l' \in \mathbb{Z}_{2L'+1}$ and $k' \in \mathbb{Z}_{2K'+1}$, we have

$$\langle \tilde{a}, v_{l'+k'(2L'+1)} \rangle_{\ell^2(\mathbb{Z}_{2N+1})} = \langle \tilde{a}, v_{k'(2L'+1)} \rangle_{\ell^2(\mathbb{Z}_{2N+1})}$$

Remark 3.1. One can prove an analogous statement for the usual Fourier transform for a function supported in a small interval in $\mathbb{R}$.

Proof. By definition, for $m \in \mathbb{Z}_{2N+1}$,

$$\langle a, v_m \rangle_{\ell^2(\mathbb{Z}_{2N+1})} = \frac{1}{\sqrt{2N+1}} \sum_{n \in \mathbb{Z}_{2K+1}} a_n e^{-2i\pi nm/(2N+1)}$$

We decompose $m = l' + k'(2L'+1)$ where $k' \in \mathbb{Z}_{2K'+1}$ and $l' \in \mathbb{Z}_{2L'+1}$ so that

$$\begin{aligned}
\langle a, v_m \rangle_{\ell^2(\mathbb{Z}_{2N+1})} &= \frac{1}{\sqrt{2K'+1}\sqrt{2L'+1}} \sum_{n \in \mathbb{Z}_{2+1}} a_n e^{-2i\pi nk'/(2K'+1)} e^{-2i\pi nl'/(2N+1)} \\
&= \frac{1}{\sqrt{2L'+1}} \left(\frac{1}{\sqrt{2K'+1}} \sum_{n \in \mathbb{Z}_{2K'+1}} a_n e^{-2i\pi nk'/(2K'+1)} \right. \\
&\quad \left. + \frac{1}{\sqrt{2K'+1}} \sum_{\mathbb{Z}_{2K'+1}} (e^{-2i\pi nl'/(2N+1)} - 1)\, a_n e^{-2i\pi nk'/(2K'+1)} \right) \\
&= \frac{1}{\sqrt{2L'+1}} \langle a, v_{k'}^{K'} \rangle_{\ell^2(\mathbb{Z}_{2K'+1})} + \frac{1}{\sqrt{2L'+1}} \langle D^{l'} a, v_{k'}^{K'} \rangle_{\ell^2(\mathbb{Z}_{2K'+1})}
\end{aligned}$$

where

- a is seen as an element of $\ell^2(\mathbb{Z}_{2K'+1})$
- $(v_{k'}^{K'})_{k' \in \mathbb{Z}_{2K'+1}}$ is the orthonormal basis of $\ell^2(\mathbb{Z}_{2K'+1})$ defined by

$$v_{k'}^{K'} = \frac{1}{\sqrt{2K'+1}} (e^{2i\pi kk'/(2K'+1)})_{k \in \mathbb{Z}_{2K'+1}}$$

- for $l' \in \mathbb{Z}_{2L'+1}$, $D^{l'}$ is the $(2K'+1) \times (2K'+1)$ diagonal matrix (acting on $\ell^2(\mathbb{Z}_{2K'+1})$) with the diagonal entries

$$d_{nn}^{l'} = \begin{cases} e^{-2i\pi nl'/(2N+1)} - 1 & \text{if } |n| \leqslant K \\ 0 & \text{if not} \end{cases}$$

Obviously,

$$\sup_{l' \in \mathbb{Z}_{2L'+1}} \|D^{l'}\|_{\mathscr{L}(\ell^2(\mathbb{Z}_{2K'+1}))} \leqslant \sup_{\substack{k \in \mathbb{Z}_{2K+1} \\ l' \in \mathbb{Z}_{2L'+1}}} |e^{-2i\pi kl'/(2N+1)} - 1| = C_{K,K'}$$

and

$$C_{K,K'} \underset{K/K' \to 0}{\sim} \pi \frac{K}{K'}$$

We define

$$\tilde{a} = \sum_{n \in \mathbb{Z}_{2N+1}} \frac{1}{\sqrt{2L'+1}} \langle a, v^{K'}_{[n]_{L'}} \rangle_{\ell^2(\mathbb{Z}_{2K'+1})} v_n$$

where $[n]_{L'} \equiv n \bmod (2L'+1)$. Then $\tilde{a}$ satisfies Lemma 3.1. Indeed as the $(v_n)_{n \in \mathbb{Z}_{2N+1}}$ form an orthonormal basis of $\ell^2(\mathbb{Z}_{2N+1})$, point 2 is obvious by the definition of $\tilde{a}$. Let us check point 1; we compute

$$\begin{aligned}
\|a - \tilde{a}\|^2_{\ell^2(\mathbb{Z}_{2N+1})} &= \sum_{n \in \mathbb{Z}_{2N+1}} \left| \frac{1}{\sqrt{2L'+1}} \langle a, v^{K'}_{[n]_{L'}} \rangle_{\ell^2(\mathbb{Z}_{2K'+1})} - \langle a, v_n \rangle_{\ell^2(\mathbb{Z}_{2N+1})} \right|^2 \\
&= \sum_{l' \in \mathbb{Z}_{2L'+1}} \frac{1}{2L'+1} \sum_{k' \in \mathbb{Z}_{2K'+1}} |\langle D^{l'} a, v^{K'}_{k'} \rangle_{\ell^2(\mathbb{Z}_{2K'+1})}|^2 \\
&= \sum_{l' \in \mathbb{Z}_{2L'+1}} \frac{1}{2L'+1} \|D^{l'} a\|^2_{\ell^2(\mathbb{Z}_{2K'+1})} \\
&\leqslant C^2_{K,K'} \|a\|^2_{\ell^2(\mathbb{Z}_{2N+1})}
\end{aligned}$$

This ends the proof of Lemma 3.1.

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